Math 270: Differential Equations Day 13 Part 2

Section 4.7: Variable-Coefficient Equations

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By variable-coefficient equation, we mean $a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$

Existence and Uniqueness of Solutions

Theorem 5. If $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0, then for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

(4) $$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) ; \quad y(t_0) = Y_0 , \quad y'(t_0) = Y_1 .$$

Section 4.7: Variable-Coefficient Equations

Example 1 Determine the largest interval for which Theorem 5 ensures the existence and uniqueness of a solution to the initial value problem

$$(t-3)\frac{d^2y}{dt^2}+\frac{dy}{dt}+\sqrt{t}\,y=\ln t\,;\quad y(1)=3\,,\quad y'(1)=-5\,.$$

Section 4.7: Variable-Coefficient Equations

A special type of DE:

Cauchy–Euler, or Equidimensional, Equations

Definition 2. A linear second-order equation that can be expressed in the form

(6) $$at^2y''(t) + bty'(t) + cy = f(t) ,$$

where a, b, and c are constants, is called a **Cauchy–Euler,** or **equidimensional, equation.**

- How do you solve a Cauchy-Euler Equation?
- We first turn our attention to the homogeneous case (only)

Section 4.7: Variable-Coefficient Equations

Solving the Homogeneous Cauchy-Euler Equation (aux. equation has distinct real roots)

DE: $at^2y'' + bty' + cy = 0$

Guess: $y = t^r$ derive r values

Section 4.7: Variable-Coefficient Equations

Solving the Homogeneous Cauchy-Euler Equation (aux. equation has distinct real roots)

DE: $at^2y'' + bty' + cy = 0$

Guess: $y = t^r$

Auxiliary Equation: $ar^2 + (b-a)r + c = 0$

Situation 1:

If r_1 and r_2 are distinct real roots of the auxiliary equation, then

$$y_1 = t^{r_1} \quad \text{and} \quad y_2 = t^{r_2}$$

Are 2 independent solutions to the homogeneous Cauchy-Euler Equation

Section 4.7: Variable-Coefficient Equations

Example 2 Find two linearly independent solutions to the equation $3t^2y'' + 11ty' - 3y = 0, \quad t > 0.$

Section 4.7: Variable-Coefficient Equations

Solving the Homogeneous Cauchy-Euler Equation (other situations)

DE: $at^2y'' + bty' + cy = 0$ Auxiliary Equation: $ar^2 + (b - a)r + c = 0$

What if the auxiliary equation has a real root of multiplicity 2 or complex roots? Derive

Situation 2:
If r is a real root of the auxiliary equation of multiplicity 2, then

$$y_1 = t^r \quad \text{and} \quad y_2 = t^r \ln t$$

Are 2 independent solutions to the homogeneous Cauchy-Euler Equation

Situation 3:
If $\alpha + \beta i$ is a complex root of the auxiliary equation, then

$$y_1 = t^{\alpha}\sin(\beta \ln t) \quad \text{and} \quad y_2 = t^{\alpha}\cos(\beta \ln t)$$

Are 2 independent solutions to the homogeneous Cauchy-Euler Equation

Section 4.7: Variable-Coefficient Equations

Example 3 Find a pair of linearly independent solutions to the following Cauchy–Euler equations for $t > 0$.

(a) $t^2y'' + 5ty' + 5y = 0$ **(b)** $t^2y'' + ty' = 0$

A Condition for Linear Dependence of Solutions

Lemma 3. If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \tag{10}$$

on an interval *I* where the functions $p(t)$ and $q(t)$ are continuous and if the Wronskian†

$$W[y_1, y_2](t) := y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is zero at *any* point t **of *I***, then y_1 and y_2 are linearly dependent **on *I***.

As in the constant-coefficient case, the Wronskian of two solutions is either identically zero or never zero on I, with the latter implying linear independence on I.

Variation of Parameters

Theorem 7. If y_1 and y_2 are two linearly independent solutions to the homogeneous equation (10) on an interval I where $p(t)$, $q(t)$, and $g(t)$ are continuous, then a particular solution to (11) is given by $y_p = v_1y_1 + v_2y_2$, where v_1 and v_2 are determined up to a constant by the pair of equations

$$y_1v_1' + y_2v_2' = 0\,,$$
$$y_1'v_1' + y_2'v_2' = g\,,$$

which have the solution

$$v_1(t) = \int \frac{-g(t)\,y_2(t)}{W[y_1, y_2](t)}\,dt\,, \quad v_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\,dt\,. \tag{12}$$

Note the formulation (12) presumes that the differential equation has been put into standard form [that is, divided by $a_2(t)$].

Section 4.7: Variable-Coefficient Equations

Reduction of Order

Theorem 8. If $y_1(t)$ is a solution, not identically zero, to the homogeneous differential equation (10) in an interval I (see page 195), then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{y_1(t)^2}\, dt \qquad (13)$$

is a second, linearly independent solution.

Example 4 Given that $y_1(t) = t$ is a solution to $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$, use the reduction of order procedure to determine a second linearly independent solution for $t > 0$.

Section 4.7: Variable-Coefficient Equations

Example 5 Find a general solution to $(\sin t)y'' - 2(\cos t)y' - (\sin t)y = 0, \quad 0 < t < \pi$.